Stokes's Kernel Modifications

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SUMMARY

The incomplete global coverage of accurate gravity measurements excludes a precise determination of the geoid using Stokes's formula. Instead, an approximate solution is used in practice that combines long-wavelength Earth Gravity Models, with local gravity. This truncation of the area produces an error that will be reducing by a modification of Stokes's kernel.

Generally, the modifications of Stokes kernel are divided into two categories: the stochastic modifications of Stokes formula interested to the statistical study of the errors of: earth global model, the gravity data and the truncation of Stokes integral to a cap around the computation point. The use of this kernel modified has been successfully applied in the determination of several high-resolution regional geoid models in different areas.

The second approach called deterministic that consists only to the minimization of the truncation error or the convergence of the errors series to zero.

In this paper we expose and compare the different kernels proposed in the geodesy literature aiming to choose the best kernel for computing the precise geoid of Algeria.

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1. INTRODUCTION

The gravimetric geoid can be computed by Stokes' formula. This formula, published by Stokes already in 1849, is one of the fundamental relationships in physical geodesy. It enables the determination of the geoid model from globally distributed gravimetric measurements.
For practical considerations. However, the area of computation is often limited to a spatial domain around the computation point. Proposed by Molodenskii et al. (1962), this truncation of the area cause an error that can be reduced by introducing a modification of Stokes' kernel.

They are two different approaches in the theory of the modification of Stokes kernel: deterministic for example Wong and Gore, Meissl, Vanicek and Kleusberg and Featherstone which endeavored to minimize truncation error. Other initiatives like Sjöberg, Wenzel modified kernel by taking into account area truncation errors, terrestrial gravity data and coefficients of global geopotential model supposedly known in advance.

2. THE LEAST SQUARES MODIFICATION OF STOKES FORMULA (LSMS)

The aim of LSMS is to reduce in a least squares sense, errors of geoid height deduced from truncation of the area, terrestrial gravity data and coefficients of global geopotential model (GGM).
The original Stokes formula for gravimetric geoid model is given by:

$$N = \frac{R}{4\pi\gamma} \iint_{\sigma} S(\psi)\, \Delta g \; d\sigma \tag{2.1}$$

Where N is geoid height, R is mean radius of the Earth, ψ geocentric angle, γ normal gravity on the reference ellipsoid, Δg gravity anomaly on the geoid, $d\sigma$ surface element of integration over a unit sphere σ and $S(\psi)$ is Stokes kernel, which in closed form is given as: [Heiskanen and Moritz, 1967]

$$S(\psi) = \frac{1}{\sin\left(\frac{\psi}{2}\right)} - 6\sin\left(\frac{\psi}{2}\right) + 1 - 5\cos\psi - 3\cos\psi \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) \tag{2.2}$$

and in spectral form as:

$$S(\psi) = \sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n(cos\psi) \tag{2.3}$$

P_n is Legendre polynomial

By taking some of the above facts into consideration, and that the observed terrestrial gravity anomaly $\Delta\hat{g}$ is limited to a cap of radius σ_0. With corresponding geocentric angle ψ_0 the GGM gravity anomaly Δg_M is known up to degree and order M, the modified Stokes provides approximate geoid height $\widetilde{N}$ as:

$$\widetilde{N} = \frac{c}{2\pi} \iint_{\sigma} S^M(\psi_0)\, \Delta\hat{g}\, d\sigma + c \sum_{n=2}^{M} s_n\ \Delta g\hat{\ }_n^M \tag{2.4}$$

Where $c = \frac{R}{2\gamma}$, $S^M(\psi)$ is the modified Stokes function and s_n are the modification parameters with the assumption that $s_0=s_1=0$. [Ellmann, 2004].

$S^M(\psi)$ is expressed as:

$$S^M(\psi) = S(\psi) - \sum_{n=2}^{M} s_n \Delta g\hat{\ }_n^M \tag{2.5}$$

The coefficients of the geopotential model have been obtained via some estimation process from satellite tracking data, containing noise, which unavoidably propagate into the computed geoid undulations. One should also consider the erroneous terrestrial gravity data within the spherical integration cap.
Then, the geoid model error is expressed by;

$$\delta N = \widetilde{N} - N = \frac{c}{2\pi} \iint_{\sigma} S^M(\psi_0)\, \varepsilon^{\mathrm{T}}\, d\sigma + c \sum_{n=2}^{M} s_n\ \varepsilon_n^M - \frac{c}{2\pi} \iint_{\sigma-\sigma_0} S^M(\psi_0)\, \Delta \mathrm{g}\, d\sigma \tag{2.6}$$

Where ε^T and ε^M are errors of the terrestrial and GGM gravity anomaly respectively. By assuming these errors are random with zero statistical expectation

$$E(\varepsilon^T) = 0 \quad \text{and} \quad E(\varepsilon^M) = 0$$

Then $$E(\delta N) = \frac{c}{2\pi} \iint_{\sigma-\sigma_0} S^M(\psi)\, \Delta g\, d\sigma = -c \sum_{n=2}^{M} Q_n^M\ \Delta g_n\ d\sigma \tag{2.7}$$

where: the truncation coefficient Q_n^M are expressed as

$$Q_n^M = Q_n - \sum_{n=2}^{M} \frac{2k+1}{2} s_k e_{nk} \tag{2.8}$$

the Molodensky's truncation coefficient Q_n are given by:

$$Q_n = \int_{\psi_0}^{\pi} S(\psi) P_n\,(cos\psi)\, sin\psi\, d\psi \tag{2.9}$$

and the Paul's coefficients e_{nk} are expressed as:

$$e_{nk} = \int_{\psi_0}^{\pi} P_n(cos\psi)P_k\,(cos\psi)\; sin\psi\; d\psi$$

Now, we take care of the bias due to truncation at the same time allowing for more parameters than the available degree of the GGM. First, by modifying the Stokes function to:

$$S^L(\psi) = S(\psi) - \sum_{n=2}^{L} \frac{2n+1}{2} s_n\, P_n(cos\psi), \qquad L \geq M \tag{2.10}$$

then taking care of truncation bias. The estimator $\widetilde{N}$ becomes

$$\widetilde{N} = \frac{c}{2\pi} \iint_{\sigma_0} S^L(\psi\quad)\, \Delta\hat{g}\; d\sigma + c \sum_{n=2}^{M} (Q_n^L + s_n)\, \Delta \hat{g}_n^M \tag{2.11}$$

Consequently, the geoid model error becomes,

$$\delta N = \frac{c}{2\pi} \iint_{\sigma} S^L(\psi)\varepsilon^T d\sigma + c \sum_{n=2}^{M} (Q_n^L + s_n)\varepsilon_n^s - c \sum_{n=M-1}^{L} (Q_n^L + s_n)\Delta g_n - c \sum_{n=L+1}^{\infty} Q_n^L \Delta g_n \tag{2.12}$$

Minimization of the geoid estimator errors is the main objective of any modification procedure. Based on the spectral form of the true geoid undulation N, the expected global mean square error (MSE) of the geoid estimator $\widetilde{N}$ can be written [Ellmann, 2001]:

$$m_{\widetilde{N}}^2 = E\left(\frac{1}{4\pi} \iint_{\sigma} \left(\widetilde{N} - N\right)^2 d\sigma\right) = c^2 \sum_{n=2}^{\infty} b_n^2 dc_n + c^2 \sum_{n=2}^{\infty} \left[(b_n^* - Q_n^L - s_n^*)^2 + \left(\frac{2}{n-1} - Q_n^L - s_n^*\right)^2 \sigma_n^2\right] \tag{2.13}$$

Since all data errors are assumed to be random and with expectations zero, the norm of the total errors is thus obtained by adding their partial contributions. The first term represents the contribution due to errors of the geopotential model. The middle term is due to truncation error and the last term accounts for the influence of erroneous terrestrial data.

Where c_n is the GGM gravity anomaly signal degree variance, which can be evaluated from Ellmann, 2001] :

$$c_n = \frac{1}{4\pi} \iint_{\sigma} (\Delta g_n)^2\, d\sigma \tag{2.14}$$

dc_n is also known as error degree variance, usually obtained from GGM potential variances $\sigma^2_{C_{nm}} and\ \sigma^2_{S_{nm}}$

$$dc_n = E\left\{\frac{1}{4\pi}\iint_\sigma (\varepsilon_n^s)^2\, d\sigma\right\} = \left(\frac{GM}{a^2}\right)^2 (n-1)^2 \sum_{m=-n}^{n}(\sigma^2_{C_{nm}} + \sigma^2_{S_{nm}}) \qquad (2.15)$$

σ_n^2 is the terrestrial anomaly error degree variances obtained by:

$$\sigma_n^2 = E\left\{\frac{1}{4\pi}\iint_\sigma (\varepsilon_n^T)^2\, d\sigma\right\} \qquad (2.16)$$

Where ε_n^T is the error of Δg_n^T

The minimum MSE of the unbiased estimator ie $\delta N^2 = min$, is attained when equation (2.13) is differentiated with respect to parameters s_n i.e $(\delta N^2)/\partial s_n$ and then set to minimum by equating to zero. This results into a system of linear equations

$$\sum_{r=2}^{L} a_{kr} s_r = h_k, \qquad k = 2,3,\ldots,L \qquad (2.17)$$

where a_{kr} and h_k are modification coefficients dependent on $Q_n, e_{nk}, c_n, dc_n, \sigma_n^2$.

The difference between many kernel modifications methods comes from the way the modification parameters s_l, $l=2,3,\ldots,L$ are realized, which are the solution to the system of linear equations (2.17). On the other hand s_l depends of the quality of the data, and the characteristics of the geopotential model used while minimizing the quadratic average of the total error.

3. DETERMINISTIC APPROACHES

The truncation of the Stokes global integral into a limited spherical cap radius has been practiced by modifying the Stokes' kernel via deterministic approach. The strategy of the deterministic approach is to reduce the effect of the far-zone gravity field while truncating the Stokes global integral to a limited spherical cap.
Since Molodensky's pioneering work [Molodensky and al., 1962] , several other authors have proposed modifications to Stokes's (1849) integral. These have been based on different criteria and can be broadly classified as deterministic modifications (like. Molodensky and al. 1962; Wong and Gore 1969; Meissl 1971; Heck and Grüninger 1987; Vanícek and Kleusberg 1987; Featherstone and al. 1998).

Therefore, the deterministic kernel modifications can be further divided into two broad categories; modifications that reduce the upper bound of the truncation error according to some prescribed norm, and modifications that improve the rate of convergence of the series expansion of the truncation error.

3.1. WONG AND GORE'S MODIFICATION

Wong and Gore modified the Stokes' kernel by removing the low-degree terms of the Legendre polynomials from the original kernel.

Equation (2.1) shows that the unmodified Stokes kernel can be expressed as a Fourier series of Legendre's polynomials from *2* to ∞. When a global geopotential model of spherical harmonic degree and order *(L-1)* is included in eq. (2.1), the low-degree terms from 2 to *(L - 1)*, inclusive, are no longer required in Stokes's integral due to the orthogonality of spherical harmonics over the sphere. [Featherstone and al., 1998]

Then, the Wong and Gore kernel is written as:

$$S^L(\psi) = S(\psi) - \sum_{n=2}^{L} \frac{2n+1}{n-1} P_n(cos\psi) \tag{3.1}$$

When $L=M$, such as M is the degree of the geopotential model, the kernel is called spheroidal Stokes kernel.

After that, the spheroidal coefficient of truncation is written in function of the coefficient truncation corresponding to the spherical kernel of Stokes and a coefficient e_{nk} which one can numerically determine using the formulas of recurrence of Paul (1973) [Chuanding and al., 1998].

$$Q_n^M(\psi_0) = Q_n(\psi_0) - \sum_{k=2}^{M} \frac{2k+1}{k-1} e_{nk}(\psi_0) \tag{3.2}$$

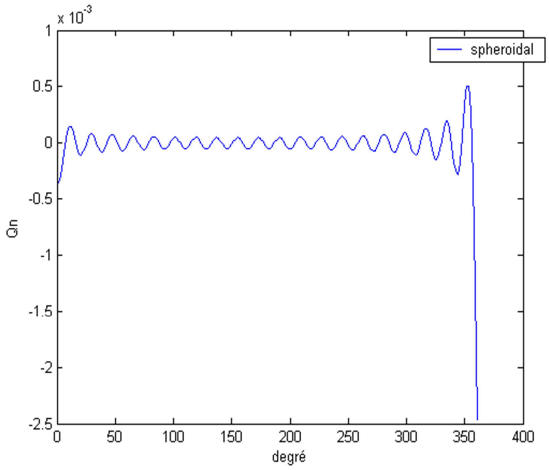

Figure 1: Spheroidal truncation coefficient

The spheroidal truncation coefficient becomes unstable in the neighborhood of degree 360 (figure 2), this divergence is to study thereafter.

3.2. VANÏCEK AND KLEUSBERG'S (1987) MODIFICATION

In a similar manner to Molodensky et al., that make a modification to the spherical Stokes kernel, Vanïcek and Kleusberg (1987) make a modification to the spheroidal Stokes kernel.

This approach minimizes the upper bound of the truncation error in a root mean squares sense.The process of minimization of the quadratic average error leads us to a system of (L-1) linear equations of the form [Vaniček and Featherstone, 1998]:

$$\begin{cases} \sum_{k=2}^{L} \frac{2k+1}{2} t_k(\psi_0) e_{nk\,k}(\psi_0) = Q_{n\ \ k}^{WG}(\psi_0) \ ; \ 2 \le n \le \mathrm{L} \\ Q_n^{WG}(\psi_0) = Q_n(\psi_0) - \sum_{k=2}^{L} \frac{2k+1}{k-1} e_{nk}(\psi_0) \end{cases} \tag{3.3}$$

$Q_n^{WG}(\psi_0)$ is the Wong and Gore truncation coefficient
The resolution of this system of linear equation (3.3) us given the coefficients t_k includes in the definition of the Vanicek and Kleusberg kernel.
Consequently, the Vanicek and Kleusberg kernel as follows:

$$S^{VK}(\psi) = S^L(\psi) - \sum_{n=2}^{L} \frac{2n+1}{2}\ t_n\, P_n(cos\psi) \tag{3.4}$$

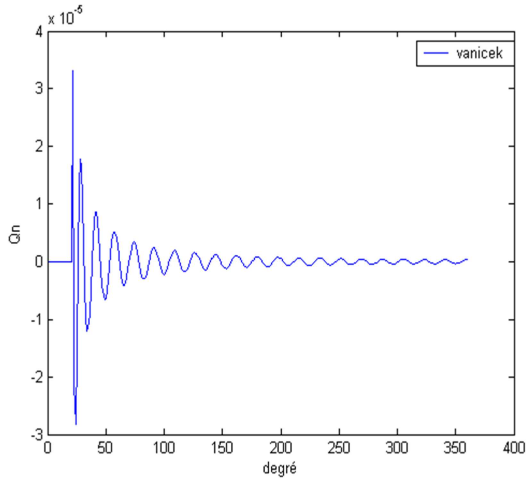

Figure 2: Truncation coefficient of Vanïcek and Kleusberg

The truncation coefficient of the Vanïcek and Kleusberg (figure 2) oscillate weakens while increasing the degree and converges thereafter towards zero for great values of degree.

3.3. MEISSL'S (1971) MODIFICATION

Meissl's modification is achieved by simply subtracting the numerical value of the spherical Stokes kernel at the truncation radius $S(\psi_0)$ from the original kernel. Thus, the Meissl modified kernel is defined as

$$S^{mei}(\psi) = \begin{cases} S(\cos\psi) - S(\cos\psi_0) & \text{si} \quad 0 \le \psi \prec \psi_0 \\ 0 & \text{si} \quad \psi_0 \le \psi \le \pi \end{cases} \tag{3.5}$$

Meissl shows that the truncation error series converge to zero faster by increasing spherical harmonic degree *n*, when the integration kernel is zero at the truncation radius ψ_0. Therefore, the effect of the truncation error on the geoid will be diminished at a greater rate when compared with an unmodified kernel. This is because the Fourier coefficients of a continuous kernel function converge to zero faster than those of a discontinuous kernel function**.** [Featherstone and al., 1998]

Therefore, the corresponding truncation error, when using the Meissl modification is:

$$Q_n^F(\psi_0) = Q_n(\psi) + \frac{S(\psi_0)}{(n+1)}\left(P_{n-1}(\cos\psi_0) - \cos\psi_0 . P_n(\cos\psi_0)\right) \tag{3.6}$$

The modification of Heck and Gruninger is similar to that of Meissl, except that the subtraction is applied, in this case, with the spheroidal kernel. The goal of this subtraction is to make continuous the function of the error spheroidal kernel.
The Heck and Gruninger kernel is given by

$$S^{HG} = \begin{cases} S^L(\cos\psi) - S^L(\cos\psi_0) & \text{si} \quad 0 \le \psi \langle \psi_0 \\ 0 & \text{si} \quad \psi_0 \le \psi \le \pi \end{cases} \tag{3.7}$$

3.4. THE HYBRID KERNEL MODIFICATION

An argument similar to that of Meissl will be used to show that this type of modification to Vanïcek and Kleusberg's kernel can reduce the truncation error still further. In this case, the integration kernel is set to zero at the truncation radius by subtracting the value of the Vanïcek and Kleusberg kernel at ψ_0. The hybrid kernel modification is given by:

$$S_L^M(cos\psi) = S^M(cos\psi) - S^M(cos\psi_0) - \sum_{k=2}^{L} \frac{2k-1}{2} t_k(\psi_0)[P_k(cos\psi) - P_k(cos\psi_0)] \tag{3.8}$$

This modification is proposed by Featherstone and al. (1998) uses a combination, where the rate of convergence of the series expansion of an already reduced truncation error by the L_2 norm is accelerated from $O(n^{-1})$ to $O(n^{-2})$ through the approach proposed by Meissl (1971).
This can be achieved either by setting the kernel to zero at the truncation radius through subtraction, or by choosing the truncation radius such that it coincides with a zero point of the kernel.

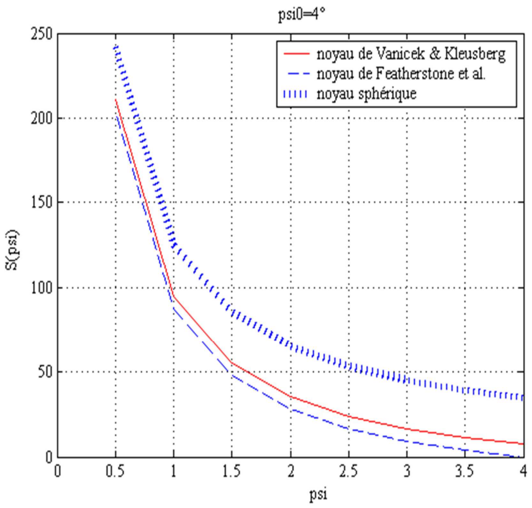

Figure3 : Featherstone and al.,Vaniček and Kleusbergand, spherical kernels for $\psi_0 = 4°$

The figure above indicates the behavior of the Featherstone and al. compared to Vaniček and Kleusberg and spherical kernels. It is noted that the Featherstone and al. kernel decrease more quickly than the other kernels and takes zero value for the spherical distance chosen.

CONCLUSION

Over recent decades two distinct groups of modification approaches have been proposed in geodetic literature.
These approaches are often called deterministic and stochastic modification methods. The deterministic approaches principally aim to reduce the effect of the neglected integration zone. However, the stochastic approach is interested to the minimization of the errors due to truncation of the integral, terrestrial data and geopotential model coefficients.

In general, all kernel modification approaches are related to each other by making some changes. In practice, the shape of the Stokes' kernel is altered so as to reduce the contribution of the residual gravity in the truncated region, to the solution of the Stokes' integral evaluated within the spherical cap of radius ψ_0 .

The stochastic modifications offer an optimal combination of two data types together with a minimization of the truncation error (in a least-squares sense), requiring reliable variance estimates of the data.

In this work we developed a theoretical study to be able to compare between the various modified kernel in order to attempt a suitable kernel for the precise determination of geoid. In addition, and because of the lack of information on the errors of the terrestrial gravity data distributed on the Algerian territory, we opt for the use of the deterministic modifications, more precisely the modified kernel by Featherstone and al., this choice is consolidated by the fact that theoretically this modification shows that the truncation error diminished at a greater rate, who allows us to give the best result in the determination of geoid undulation with a minimization of the truncation error.

REFERENCES

Chuaning. Z, Zhonglian. L, Xiaoping. W, 1998; Truncation error formulae for disturbing gravity vector, Journal of geodesy, 72:119 - 123.

Ellmann. A, 2001; Least square modification of Stokes formula with application to the Estonian geoid, Geodesy report No. 1056, Royal Institute of Technology (Kungl Tekniska HOGSKOLAN).

Ellmann. A, 2004, The geoid for the Baltic countries determined by the least squares modification of Stokes' formula]. Doctoral Dissertation in geodesy report No. 1061, Royal Institute of Technology (Kungl Tekniska HOGSKOLAN).

Featherstone. W.E, Evans. J.D, Olliver J. G, 1998; A Meissl–modified Vaniček and Kleusberg kernel to reduce the truncation error in gravimetric geoid computations.
Journal of geodesy, 72: 154- 160.

Heiskanen. W. A, Moritz. H, 1967; Physical Geodesy. W.H Freeman & company.San Francisco 1967.

Molodenskii. M.S, Eremeev. V.F, Iyurkina. M, 1962; Methods for study of the Externel Gravitational Field and Figure of the earth. Israel programme for the Translation of the Scientific Publications.

Vaniček. P, Featherstone. W.E, 1998; Performance of three types of Stokes' kernel in the combined solution for the geoid, Journal of geodesy, 72: 684- 697.

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